

On Mimicking Networks Representing Minimum Terminal Cuts

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Abstract

Given a capacitated undirected graph $G = (V, E)$ with a set of terminals $K \subset V$, a *mimicking network* is a smaller graph $H = (V_H, E_H)$ that exactly preserves all the minimum cuts between the terminals. Specifically, the vertex set of the sparsifier V_H contains the set of terminals K and for every bipartition $U, K - U$ of the terminals K , the size of the minimum cut separating U from $K - U$ in G is exactly equal to the size of the minimum cut separating U from $K - U$ in H .

This notion of a *mimicking network* was introduced by Hagerup, Katajainen, Nishimura and Ragde [HKNR95] who also exhibited a mimicking network of size 2^{2^k} for every graph with k terminals. The best known lower bound on the size of a mimicking network is linear in the number of terminals. More precisely, the best known lower bound is $k + 1$ for graphs with k terminals [CSWZ00].

In this work, we improve both the upper and lower bounds reducing the doubly-exponential gap between them to a single-exponential gap. Specifically, we obtain the following upper and lower bounds on mimicking networks:

- Given a graph G , we exhibit a construction of mimicking network with at most $(|K| - 1)$ 'th Dedekind number ($\approx 2^{\binom{k-1}{\lfloor (k-1)/2 \rfloor}}$) of vertices (independent of size of V). Furthermore, we show that the construction is optimal among all *restricted mimicking networks* – a natural class of mimicking networks that are obtained by clustering vertices together.
- There exists graphs with k terminals that have no mimicking network of size smaller than $2^{\frac{k-1}{2}}$.

We also exhibit improved constructions of mimicking networks for trees and graphs of bounded tree-width.

keywords: Approximation algorithms, Graph algorithms, Vertex sparsification, Cut sparsifier, Mimicking networks, Terminal cuts, Realizable external flow, Network flow.

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1 Introduction

Suppose there are small number of terminals or clients that are part of a huge network such as the internet. Often, it is useful to construct a smaller graph which preserves the properties of the huge network that are relevant to the terminals. For example, if the terminals or clients are interested in routing flows through the large network, one would want to construct a small graph which preserves the routing properties of the original network. The notion of *mimicking networks* introduced by Hagerup et. al. [HKNR95] is an effort in this direction.

Let G be an undirected graph with edge capacities c_e for $e \in E$, and a set of k terminals $K \subset V$. A *mimicking network* for G is an undirected capacitated graph $H = (V_H, E_H)$ such that $K \subseteq V_H$ and for each subset $U \subset K$ of terminals, the size of the minimum cut separating U from $K - U$ in H is exactly equal to the size of the minimum cut separating U and $K - U$ in the graph G . As a corollary, the set of realizable external flows (possible total flows at terminals) in G are preserved in a mimicking network. Therefore, the smaller graph H *mimics* the graph G in terms of external flows routable through it. The vertices of the mimicking network that are not terminals, namely $V_H - K$ will be referred to as *Steiner* vertices.

The work of Hagerup et. al. [HKNR95] exhibited a construction a mimicking network with at most 2^{2^k} vertices for every graph with k terminals. Subsequently, Chaudhuri et. al. [CSWZ00] proved that there exists graphs that require at least $(k + 1)$ vertices in its mimicking network. The same work also obtained improved constructions of mimicking networks for special classes of graphs namely, bounded treewidth and outer planar graphs. Specifically, they showed that graphs of treewidth t admit a mimicking network of size $k2^{2^{3(t+1)}}$, while outerplanar graphs admit mimicking networks of size $(10k - 6)$.

Mimicking networks constituted the main building block in the development of $O(n)$ time algorithm for computing maximum $s - t$ flow in a bounded treewidth network [HKNR95] and for obtaining an optimal solution for the all-pairs minimum-cut problem in the same class of networks [ACZ98]. However, there still remained a doubly exponential gap between the known upper and lower bounds for the size of mimicking networks for general graphs.

1.1 Vertex Sparsifiers

Closely tied to mimicking networks is the more general notion of *vertex sparsifiers* introduced by Moitra [Moi09]. Roughly speaking, a *vertex cut sparsifier* is a mimicking network that only approximately preserves the cut values. Formally, let G be an undirected graph with edge capacities c_e for $e \in E$, and a set of k terminals $K \subset V$. A *vertex cut sparsifier* with quality q is an undirected capacitated graph $H = (V_H, E_H)$ such that $K \subseteq V_H$ and for each subset $U \subset K$ of terminals, the size of the minimum cut separating U from $K - U$ in H is within a factor q of the size of the minimum cut separating U and $K - U$ in the graph G .

The original motivation behind the notion of vertex cut sparsifiers was to obtain improved approximation algorithms for certain graph partitioning and routing problems. If the solution to some combinatorial optimization problem only depends on the values of the minimum cuts separating terminal subsets, then given any approximation algorithm for the problem, we can first compute a cut sparsifier H for graph G and run the approximation algorithm on the graph H instead of G . If the approximation guarantee of the algorithm depended on the number of the vertices of the input graph, then this would yield an algorithm whose approximation guarantee only depends on the size of the sparsifier H .

The problem of constructing *vertex cut sparsifiers* has received considerable attention since their introduction in [Moi09]. Naturally, the goal would be to obtain as good an approximation as possible, while keeping the size of the sparsifier H small. In fact, the notion of vertex sparsifiers as defined in [Moi09] require that the graph H have only the terminals K as the vertices, i.e., $V_H = K$ (no Steiner vertices). Much of the subsequent efforts have been focused on vertex sparsifiers with this additional requirement that $V_H = K$. In this setting, Moitra [Moi09] showed the existence of vertex sparsifiers with quality $O(\log^2 k / \log \log k)$. Subsequent works by Leighton et al. [LM10], Englert et al. [EGK⁺10] and Makarychev et al. [MM10] gave polynomial-time algorithms for constructing $O(\log k / \log \log k)$ cut sparsifiers, matching the best known existential upper bound. On the negative side, Leighton and Moitra [LM10] showed a lower bound of $\Omega(\log \log k)$ on the quality of cut sparsifiers without Steiner vertices, which was subsequently improved to $\Omega(\sqrt{\log k / \log \log k})$ [MM10].

In light of these lower bounds, it is natural to wonder if better approximation guarantees could be obtained by vertex sparsifiers that include *steiner vertices*, i.e., vertices of the sparsifier H are a strict super-set of the set of terminals K . In fact, for $k \geq 4$, there exist graphs for which no cut sparsifier without Steiner vertices preserves terminal cuts exactly. But by Hagerup [HKNR95], there exists cut sparsifiers (mimicking networks) with 2^{2^4} nodes that exactly preserves all the cuts.

Initiating the study of vertex sparsifiers with steiner nodes, Chuzhoy [Chu12] exhibited efficient algorithms to construct $3(1 + \epsilon)$ -quality cut sparsifiers of size $O(C/\epsilon)^3$ for a constant $\epsilon \in (0, 1)$, where C denotes the total capacity of the edges incident on the terminals, normalized so as to make all the edge-capacities at least 1. The same work also gives an efficient construction of a $(68 + \epsilon)$ -quality vertex flow sparsifier of size $C^{O(\log \log C)}$ in time $n^{O(\log C)} \cdot 2^C$. Notice that the size of the sparsifiers depend on the total capacity C of edges incident at the terminals, which could be arbitrarily large compared to the number of terminals k .

While there has been progress in efficient constructions of vertex sparsifiers without Steiner nodes, the power of vertex sparsifiers with Steiner nodes is poorly understood. For instance, the following question originally posed by Moitra [Moi09] remains open.

Do there exists cut sparsifiers with $k^{O(1)}$ additional steiner nodes that yield a better than $O(\log k / \log \log k)$ approximation?

In fact, Moitra [Moi09] points out that there could exist exact cut sparsifiers (quality 1) with only k additional Steiner nodes.

1.2 Our results:

In this paper, we show upper and lower bounds for mimicking networks aka vertex cut sparsifiers with quality 1. First, we present an improved bound on the size of mimicking networks for general graphs. Specifically, we exhibit a construction of mimicking networks with at most $(|K| - 1)$ 'th Dedekind number ($\approx 2^{\binom{k-1}{\lfloor (k-1)/2 \rfloor}}$) of vertices, as opposed to 2^{2^k} vertices.

Theorem 1.1. For every graph G , there exists a mimicking network with quality 1 that has at most $(|K| - 1)$ 'th Dedekind number ($\approx 2^{\binom{k-1}{\lfloor (k-1)/2 \rfloor}}$) vertices. Further, the mimicking can be constructed in time polynomial in n and 2^k .

We also note that the mimicking network constructed above is a *contraction-based* in the sense that the mimicking network H is constructed as follows: Fix an appropriate partition \mathcal{C} of the vertices of the graph G and contract every subset of vertices $S \in \mathcal{C}$ in the partition to form a

vertex of H . Contraction-based sparsifiers have also referred to as *restricted sparsifiers* in literature [CLLM10] who show that they are a strictly stronger notion than vertex cut sparsifiers. For restricted sparsifiers, we will use the terms – non-terminal and Steiner vertex interchangeably.

We prove that construction is optimal for the class of contraction-based mimicking networks.

Theorem 1.2. Let G be a graph with unique minimum terminal cuts. Then the mimicking network constructed using Algorithm 1 is an optimal among contraction-based mimicking networks for G i.e., it has minimum number of vertices among all contraction-based mimicking networks.

Next, we obtain an exponential lower bound on the size of the mimicking networks. Specifically, we show the following result.

Theorem 1.3. There exists graphs G for which every mimicking network has size at least $2^{(k-1)/2}$.

We also obtain improved constructions of mimicking networks for special classes of graphs like trees and graphs of bounded tree width. For the case of a tree, we show that $\frac{13|K|}{8} - \frac{3}{2}$ suffice, while for a graph with treewidth t there exists mimicking networks of size $|K|2^{\binom{3t+2}{\lfloor (3t+2)/2 \rfloor}}$. We also exhibit mimicking networks that preserve cuts separating terminal set of size ≤ 2 from other terminals using only one extra Steiner vertex.

Related Work In an independent work, Krauthgamer and Rika [KR12] obtained upper and lower bounds for the size of mimicking networks in general graphs, and certain special classes of graphs. Specifically, they show a lower bound of $2^{\Omega(k)}$ for the size of mimicking networks even for the case of bipartite graphs. Furthermore, the lower bound is shown to hold for the size of any data structure that stores all the minimum terminal cut values of a graph. The paper also obtains improved upper and lower bounds for the special case of planar graphs.

It has been brought to our attention that the improved upper bound of Dedekind number of vertices for mimicking networks was also observed by Chambers and Eppstein [CE10].

2 Preliminaries

In this section, we set up the notation and present formal definitions of vertex cut sparsifiers and mimicking networks. Let $G = (V, E)$ be an undirected capacitated graph with edge capacities $c(e)$ for all edges $e \in E$ and a set $K \subset V$ of terminals of size k . Without loss of generality, we assume that G is connected, otherwise each component can be handled separately. Let $c : E \rightarrow \mathbb{R}^+$ be the capacity function of the graph. Let $h_G : 2^V \rightarrow \mathbb{R}^+$ denote the cut function of G :

$$h_G(A) = \sum_{e \in \delta(A)} c(e)$$

where $\delta(A)$ denote the set of edges crossing the cut $(A, V \setminus A)$. Now we define terminal cut function $h_K^G : 2^K \rightarrow \mathbb{R}^+$ on K as

$$h_K^G(U) = \min_{A \subset V, A \cap K = U} h_G(A)$$

In words, $h_K^G(U)$ is the cost of the minimum cut separating U from $K \setminus U$ in G . Let $S(U)$ be the smallest subset of V such that $h_G(S(U)) = h_K^G(U)$, $S(U) \cap K = U$ i.e., $S(U)$ is the partition containing U in the minimum terminal cut separating U from $K - U$ and if there are multiple

minimum terminal cuts we take any one with minimum number of vertices in the partition that contains U . For any fixed $U \subset K$, the minimum cut $h_K^G(U)$ can be computed efficiently. We will sometimes abuse the notation and use $h_K^G(U)$ to denote both the size of the minimum cut and the set of edges belonging to the minimum terminal cut.

If $|U| = 1$, we call the minimum terminal cut separating U from $K - U$ to be *mono-terminal cut*. If $|U| \leq 2$, we call the minimum terminal cut separating U from $K - U$ to be *bi-terminal cut*.

Definition 2.1. $H = (V_H, E_H)$ is a cut-sparsifier for the graph $G = (V, E)$ and the terminal set K , if $K \subseteq V_H$ and if the cut function $h_K^H : 2^{V_H} \rightarrow \mathbb{R}^+$ of H satisfies for all $U \subset K$,

$$h_K^G(U) \leq h_K^H(U).$$

Quality of cut sparsifier is a measure of how well the cut function of H approximates the terminal cut function.

Definition 2.2. The quality of a cut sparsifier H : $Q_C(H)$ is defined as

$$\max_{U \subset K} h_K^H(U) / h_K^G(U).$$

In this paper, we will study mimicking networks that are a special class of vertex sparsifiers.

Definition 2.3. A vertex sparsifier H for graph G and terminal set K is a mimicking network if $Q_C(H) = 1$.

Nearly all existing constructions of vertex sparsifiers are based on edge-contractions. Now we present a simple lemma to show contraction of edges always gives us a vertex sparsifier.

Lemma 2.4. Given a graph G and an edge e , contracting the edge e in the graph G will not decrease the value of any minimum terminal cut. [Moi]

Proof. Let G/e be the graph obtained by contracting the edge $e = (u, v)$ in the graph G . For any $U \subset K$, the minimum cut in G/e separating U from $K - U$ is also a cut in G separating U from $K - U$, with the additional restriction that u and v appear on the same side of the cut. Thus contracting an edge (whose endpoints are not both terminals) cannot decrease the value of minimum cut separating U from $K \setminus U$ for $U \subset K$. \square

Vertex sparsifiers that can be obtained by contracting edges of the original graph will be referred to as *contraction-based* vertex sparsifiers.

Definition 2.5. A graph $H = (V_H, E_H)$ is a *contraction-based* vertex sparsifier/mimicking network of graph $G = (V, E)$ with terminal set K if there exists a function $f : V \rightarrow V_H$ such that the edge cost function of H is defined as follow: $c_H(y, z) = \sum_{u, v | f(u)=y, f(v)=z} c(u, v)$ where $(y, z) \in E(H)$ and $(u, v) \in E(G)$.

3 Improved Upper Bounds on Size of Mimicking Networks

In this section we construct a mimicking network for a given undirected, capacitated graph $G = (V, E)$ with a set of terminals $K (\subset V) := \{v_1, v_2 \dots v_k\}$. Without loss of generality, we may assume G to be connected, otherwise we can consider each component separately.

Theorem 3.1. (*Restatement of Theorem 1.1*) For every graph G , there exists a mimicking network with quality 1 that has at most $(|K| - 1)$ 'th Dedekind number ($\approx 2^{\binom{k-1}{\lfloor (k-1)/2 \rfloor}}$) vertices. Further, the mimicking network can be constructed in time polynomial in n and 2^k .

Proof. First we present the algorithm 1 that constructs the mimicking network from the graph.

<p>input : A capacitated undirected graph G, set of terminals $K \subset V$ output: A capacitated undirected graph H.</p> <ol style="list-style-type: none"> 1 Find all $2^{k-1} - 1$ minimum terminal cuts using max-flow algorithm; 2 Partition the graph into $2^{2^{k-1}-1}$ clusters $\mathcal{C}_1, \mathcal{C}_2 \dots \mathcal{C}_{2^{2^{k-1}-1}}$ such that two vertices u, v belong to same cluster if they appear on same side of all the minimum terminal cuts ; 3 Contract each non-empty cluster into single node ; 4 Return the contracted graph H ;
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Algorithm 1: ALGORITHM TO CONSTRUCT EXACT-CUT-SPARSIFIER

We claim that H exactly preserves all minimum terminals cuts.

Claim 3.2. H is a mimicking network for G .

Proof of claim: Note that we are just mapping vertices of G to vertices of H and not deleting any edges of G in H , thus the minimum cut value can only grow up. Hence, $h_K^G(U) \leq h_K^H(U)$ for any $U \subset K$. But the minimum cut separating U from $K - U$ in H is the dictator cut parallel to the i 'th axis. It contains only the edges of the minimum cut separating U and $K - U$ in G . Thus $h_K^G(U) \geq h_K^H(U)$. Therefore we get $h_K^G(U) = h_K^H(U)$. \square

We upper bound the number of vertices in H by Dedekind number to complete the proof. \square

While the algorithm creates $2^{2^{k-1}} - 1$ clusters, we will argue that by an appropriate choice of cuts many of the clusters will be empty. Let $N(k)$ be the number of vertices in the mimicking network constructed by Algorithm 1, i.e., it is the number of non-empty regions created by $2^{k-1} - 1$ minimum terminal cuts. Here we show $N(k)$ is at most $(k - 1)$ 'th Dedekind number. Dedekind numbers are a rapidly-growing integer sequence defined as follows: Consider the partial order \subseteq induced on the subsets of an n -element set by containment. The n 'th Dedekind number $M(n)$ counts the number of antichains in this partial order. Equivalently, it counts monotonic Boolean functions of n variables, the number of elements in a free distributive lattice with n generators, or the number of abstract simplicial complexes with n elements

For a terminal cut $[U, K - U]$ where $v_k \notin U$, let $\{S(U), V_G - S(U)\}$ denote the partition induced by the minimum cut separating $[U, K - U]$. If there are multiple minimum terminal cuts we take any one with smallest cardinality $|S(U)|$. Now let us prove two structural properties of these minimum terminal cuts.

Lemma 3.3. If $X \subseteq Y \subseteq K$ then $S(X) \subseteq S(Y)$.

Proof. From submodularity property of cuts we get,

$$\begin{aligned}
(h_G(S(X)) + h_G(S(Y))) &\geq (h_G(S(X) \cup S(Y)) + h_G(S(X) \cap S(Y))) \\
&\geq (h_G(S(X \cup Y)) + h_G(S(X \cap Y))) = (h_G(S(Y)) + h_G(S(X))). \quad (1)
\end{aligned}$$

Here the second inequality follows from the fact $h_G(S(X) \cup S(Y)) \geq h_G(S(X \cup Y))$ and $h_G(S(X) \cap S(Y)) \geq h_G(S(X \cap Y))$. Now as all inequalities are tight in (1), we get $h_G(S(X) \cup S(Y)) = h_G(S(X \cup Y)) = h_G(S(Y))$ and $h_G(S(X) \cap S(Y)) = h_G(S(X \cap Y)) = h_G(S(X))$. We have $h_G(S(X) \cap S(Y)) = h_G(S(X))$, but recall that among all minimum cuts separating $(X, K - X)$, $S(X)$ has the smallest cardinality. This implies $S(X) \subseteq S(Y)$. \square

Lemma 3.4. If $X \cap Y = \phi$ then $S(X) \cap S(Y) = \phi$.

Proof. Assume $S(X) \cap S(Y) \neq \phi$. Then $h_G(S(X) \setminus S(Y)) + h_G(S(Y) \setminus S(X)) \leq (h_G(S(X)) + h_G(S(Y)))$. On the other hand as we always take the minimum terminal cut with smallest $|S(X)|$. Hence $h_G(S(X)) < h_G(S(X) \setminus S(Y))$ and $h_G(S(Y)) < h_G(S(Y) \setminus S(X))$. This contradicts. \square

Note that each region created by algorithm 1, is basically intersection of partitions containing $S(X)$ for some minimum terminal cuts $(X, K - X)$ and complement of $S(X)$ for remaining minimum terminal cuts. Let $X \subseteq \{U \subset K, v_k \notin U\}$ i.e., X is a collection of subsets of K that do not contain v_k . Let us define $A(X) = (\cap_{Z \in X} S(Z)) \cap (\cap_{W \notin X} \overline{S(W)})$. Each $A(X)$ corresponds to a cluster produced by the algorithm. We will show that $A(X)$ is empty for many choices of X .

Lemma 3.5. If $A(X) \neq \phi$ then X is upward closed set i.e., $(\forall P \in X, P \subseteq Q \Rightarrow Q \in X)$.

Proof. Suppose there exists a $Q \notin X$ such that for some $P \in X$ and $Q \supseteq P$. From lemma 3.3,

$$S(P) \subseteq S(Q). \quad (2)$$

Also, note that by definition, $A(X) \subseteq S(P) \cap \overline{S(Q)}$. Hence, we get $A(X) \subseteq S(P) \cap \overline{S(Q)} = \phi$ - a contradiction. \square

From lemma 3.5, if $A(X) \neq \phi$ then X is upward closed set. Now minimal elements of upper sets form an antichain. So $N(k)$ is upper bounded by the number of antichains of subsets of an $(k - 1)$ -element set i.e., $M(k - 1)$. Kleitman and Markowsky[KM] had shown that:

$$\binom{n}{\lfloor n/2 \rfloor} \leq \log M(n) \leq \binom{n}{\lfloor n/2 \rfloor} (1 + O(\log n/n)) \quad (3)$$

Moreover from lemma 3.4, if there are two completely disjoint elements in X that will lead to an empty region. So $N(k)$ is upperbounded by the number of antichains of subsets of $(k - 1)$ -element sets where all members of the antichain share at least one common element. Let us call this number to be $Z(k - 1)$. Clearly $M(k - 2) \leq Z(k - 1) \leq M(k - 1)$. Table 1 compares different bounds.

The observations made in this section together with results on bounded treewidth on [CSWZ00] implies improved bound for graphs with bounded treewidth.

Corollary 3.6. Let G be a n -vertex network of treewidth t . Then we can create an mimicking network for G that has size at most $k 2^{\binom{3(t+1)}{\lfloor 3(t+1)/2 \rfloor}}$.

3.1 Contraction-Based Mimicking Networks

Here we will show that on every graph G that has unique minimum terminal cuts, Algorithm 1 produces a mimicking network that is optimal among all contraction-based mimicking networks.

Table 1: Different bounds related to $N(k)$

k	Lower bound	Best Upper bound	Upper Bound from Contraction	$(k-1)$ th Dedekind No.	$2^{2^{k-1}} - 1$
			$Z(k-1)$	$M(k-1)$	
2	2	2	2	2	3
3	3	3	4	5	15
4	5	5	11	19	255
5	6	6	54	167	65535
6	9	*	687	7580	4.29×10^9

Theorem 3.7. (*Restatement of Theorem 1.2*) Let G be a graph with unique minimum terminal cuts. Then the mimicking network constructed using Algorithm 1 is optimal among contraction-based mimicking networks for G i.e., it has minimum number of vertices among all contraction-based mimicking networks.

Proof. Let H be the contraction-based mimicking network for graph G with terminal set K constructed using function $\phi : V(G) \rightarrow V_H$ in Algorithm 1. First, we claim that all edges in H belong to some minimum terminal cut in G .

Claim 3.8. For all edges $(y, z) \in G$, $\phi(y) \neq \phi(z)$ if and only if $(y, z) \in h_K^G(U)$ for some $U \subset K$.

Proof. The claim is clear from the construction presented in Algorithm 1. Two vertices are merged if and only if the edge between them does not belong to any minimum cut. \square

Assume H' is the optimal contraction-based mimicking network with minimum number of vertices, i.e., $|V(H')| \leq |V_H|$. Since H' is contraction-based, it is defined by a function $\phi' : V(G) \rightarrow V_{H'}$.

Claim 3.9. For all edges $(y, z) \in G$, if $(y, z) \in h_K^G(U)$ for some $U \subset K$ then $\phi'(y) \neq \phi'(z)$

Proof. Consider an $e = (y, z)$ in the original graph G , that belongs to some minimum terminal cut $(U, K - U)$. We claim that the clusters containing y and z are distinct in H' .

By definition of H' , the minimum cut $h_K^{H'}(U)$ has the same value as the minimum terminal cut $h_K^G(U)$. Since all minimum terminal cuts in G are unique, this implies that the cut induced by $h_K^{H'}(U)$ in G is exactly the same as $h_K^G(U)$. Therefore, for every edge (y, z) in the graph G that belongs to a minimum terminal cut $h_K^G(U)$, the corresponding clusters in H' are distinct. \square

From the previous two claims, $\phi(y) \neq \phi(z) \implies \phi'(y) \neq \phi'(z)$ for every edge $(y, z) \in G$. This implies that the number of clusters in H is at most the number of clusters in H' . \square

4 Exponential Lower bound

In this section we will exhibit lower bounds on the size of mimicking networks using a subtle rank argument. Fix $p = 2^{k-1} - 1$ for the remainder of the section.

Definition 4.1. A *minimum terminal cut vector (MTCV)* $m^{G,K}$ for graph G with terminal set K is a p -dimensional vector where i 'th coordinate $m_i^{G,K} = h_K^G(U_i)$ i.e., it corresponds to the value of terminal cut separating i 'th subsets of terminals from rest of the terminals for $i \in \{1, 2, \dots, p (= 2^{k-1} - 1)\}$.

Let M_k be the set of all possible minimum terminal cut vectors with k terminals. Not all vectors $v \in \mathbb{R}^{2^{k-1}-1}$ can be minimum terminal cut vectors. The submodularity of the cut function introduces constraints on the coordinates of the minimum terminal cut vector. For example there are 3 possible terminal cuts for graphs with terminal set size 3. However $[0.1, 0.1, 0.8]$ is not a valid MTCV. First we prove that these minimum terminal cut vectors form a convex set.

Lemma 4.2. M_k is a convex cone in $\mathbb{R}^{2^{k-1}-1}$.

Proof. Note that by scaling the edges of a graph G , the corresponding minimum terminal cut vector also scales. Therefore, it is sufficient to show the convexity of the set M_k .

Let G_1 and G_2 be graphs with terminal set K of size k . Let N_1 and N_2 be their set of non-terminals respectively i.e., $N_i \cup K = V(G_i)$ for $i = 1, 2$. Note that these graphs might have different edge weights or different number of vertices. So depending on the edge values minimum terminal cuts will have different values. Let us assume t_1 and t_2 be the minimum terminal cut vectors for graphs G_1 and G_2 with same terminal set K and non negative edge cost functions \mathcal{C}_1 and \mathcal{C}_2 respectively. We claim that for any nonnegative λ_1, λ_2 such that $\lambda_1 + \lambda_2 = 1$, there exists a graph H with same terminal set K and edge cost function \mathcal{C}' such that its minimum terminal cut vector $t' = \lambda_1 t_1 + \lambda_2 t_2$. We create H with nonterminals $N_1 \cup N_2$. We start with all edge costs in H to be 0. Then for $i = 1$ and 2, for all edges $(u, v) \in G_i$, we increase the cost of edge (u, v) in H by $\lambda_i \mathcal{C}_i(u, v)$. The final graph has a minimum terminal cut vector of value $\sum_{i=1}^2 \lambda_i t_i$. \square

Now we show the central lemma regarding the range of the minimum terminal cut vectors.

Lemma 4.3. The set M_k has nonzero volume.

Proof. The $\mathbf{0}$ vector is MTCV for a completely disconnected graph. For each $i \in \{1, \dots, 2^{k-1} - 1\}$, we will show that a line segment in the i^{th} direction belongs to M_k . By the convexity of the set M_k (lemma 4.2) this will imply that the set M_k has nonzero volume, i.e., full dimensional.

To demonstrate a line segment along direction $i \in \{1, \dots, 2^{k-1} - 1\}$, we will show that there exist two MTCVs which differ only in i 'th coordinate and same in all other $p - 1$ coordinates. Fix a subset U_i of terminals. To construct MTCVs that differ only on the i^{th} coordinate, construct a graph H_i for terminal sets U_i as shown in Fig. 1. Add all terminals in $K - U_i$ to a non-terminal u_0 with edge costs $1/|K - U_i|$. Add all terminals in U_i to another non-terminal v_0 with edge costs $1/|U_i|$. Put an edge between u_0 and v_0 with edge cost $1 - \epsilon$ where $0 < \epsilon < \min\{1/|U_i|, 1/|K - U_i|\}$. So, value of minimum terminal cut separating U_i from $K - U_i$ is $1 - \epsilon$ and it contains only the edge (u_0, v_0) . All other terminal cuts have value ≤ 1 and does not contain the edge (u_0, v_0) . So, we can change value of ϵ between 0 and $\min\{1/|U_i|, 1/|K - U_i|\}$ to obtain a line segment contained in M_k along direction i . \square

Definition 4.4. For a given graph G with terminal set K , the cut matrix S_G is a $p \times |E(G)|$ matrix where $S_{ij} = 1$ if edge $e_j \in h_K^G(U_i)$ and 0 otherwise.

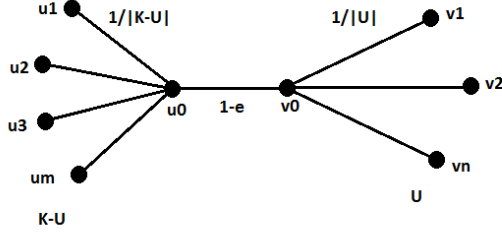


Figure 1: Graph corresponding to terminal cut $[U, K_U]$

Theorem 4.5. (*Restatement of Theorem 1.3*) There exists graphs G for which every mimicking network has size at least $2^{(k-1)/2}$.

Proof. Suppose every graph G with k terminals has a mimicking network with t vertices.

Consider a mimicking network H with t vertices for a graph G with k terminals. There are $2^t - 1$ possible cuts in the graph H . Therefore, there are at most $(2^t - 1)^p$ different cut matrices S_H of H . The specific cut matrix S_H depends on the weights of the edges in H .

Let us refer to these matrices as $S_1, S_2, \dots, S_{(2^t-1)^p}$. Each matrix S_i can be thought of as a linear map $S_i : \mathbb{R}^{\binom{t}{2}} \rightarrow \mathbb{R}^{2^{k-1}-1}$. For every graph G , there exists a choice of weights w_{ij} for the edges of H , and a choice of cut matrix S_ℓ (determined by the weights), such that $S_\ell w$ is equal to the minimum terminal cut vector h_K^G of the graph G . Therefore, the set M_k of all MTCVs is in the union of the ranges of the linear maps $\{S_i\}_{i=1}^{(2^t-1)^p}$.

However, since M_k has non-zero volume (is of full dimension), at least one of the linear maps S_i must have rank $= 2^{k-1} - 1$. Therefore $\binom{t}{2} \geq 2^{k-1} - 1$ implies that $t \geq 2^{(k-1)/2}$. \square

Corollary 4.6. There exists graphs G for which every cut sparsifier that preserves C minimum terminal cuts exactly has size at least $|C|^{1/2}$.

As the graph constructed in the theorem 1.3 has tree-width $(k+1)$, we get the following corollary.

Corollary 4.7. There exists graphs G with treewidth $\geq (k+1)$ for which every mimicking network has size at least $2^{(k-1)/2}$.

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A Improved Constructions for Special Classes of Graphs

A.1 Trees

Theorem A.1. Given an undirected, capacitated tree $T = (V, E)$ and a set $K \subset V$ of terminals of size k , we can construct a mimicking network $T_H = (V_H, E_H)$ for which the cut-function exactly approximates the value of *every* minimum cut separating any subset U of terminals from the remaining terminals $K - U$ where $|V_H| \leq 2k - 1$ and this is tight for contraction-based mimicking networks. We can also create an outerplanar mimicking network which has at most $\frac{13k}{8} - \frac{3}{2}$ vertices.

Proof. Let H' be the smallest sized mimicking network. We can assume each non-leaf non-terminal vertex in H' has degree at least 3. Otherwise, if nonterminal vertex v is a degree 2 vertex with neighbor u and w , then we can delete v and add an edge (u, w) with cost $\min(c(v, u), c(v, w))$ to preserve the minimum terminal cuts. In other words, we can contract the minimum capacity edge among (v, u) and (v, w) . Similarly if a nonterminal is a leaf, we can delete that nonterminal as it does not affect any minimum terminal cuts. Therefore, finally the tree T' contains only terminals as leaves and each non-leaf vertex has degree at least 3. So at most there are $(2k - 2)$ vertices.

To show this is tight for contraction-based mimicking network, consider a 3-regular tree with uniform edge costs and leaves as terminals. Each edge e is in at least one unique minimum terminal cut C_e . To preserve cut C_e , we can not contract e . Thus we need at least $(2k - 3)$ edges in this case.

Now we add appropriate 0-cost edges(if needed) in T' to make the tree 3-regular and set of terminals as set of leaves. We call this tree T . We can rearrange the tree such that for any node v in tree T , height of the subtree rooted at left child of v is greater than the height of the subtree rooted at right child of v . Now we define an operation called $(Y-\Delta)$ -transformation which reduces the number of vertices further. However the mimicking network remains no more contraction-based. Let x be a degree-3 nonterminal with neighbors u, v, w , then we can delete x and add edges $(u, v), (v, w), (w, u)$ with edge cost $\frac{c(u,x)+c(v,x)-c(w,x)}{2}, \frac{c(v,x)+c(w,x)-c(u,x)}{2}, \frac{c(u,x)+c(w,x)-c(v,x)}{2}$ respectively. We call this $(Y-\Delta)$ -transformation. We consider non-terminals one by one in an in-order traversal of T . We apply the transformation if a vertex has degree-3 and modify the graph. Then we find the next vertex in the in-order traversal of T that has degree 3 in the modified graph. If there exists such a vertex, we continue applying the transformation on it. Otherwise we stop to get the mimicking network H . Note that H is a cactus graph i.e., two cycles share at most one vertex in the graph. This is also an outerplanar graph. Assume $V(H) = n$. Now we claim that there are at most $\lfloor k/2 \rfloor$ leaves in H . Consider the leaves(terminals) in the in-order traversal v_1, v_2, \dots, v_k . Pair (v_i, v_{i+1}) for $i = 1, 2, \dots, \lfloor k/2 \rfloor$. We claim that at most one of them is a leaf after completion of $(Y-\Delta)$ -transformation. Take the path from v_i to v_{i+1} in T . At least one degree-3 nonterminals v_t is on the path such that $(Y-\Delta)$ -transformation was applied to v_t , making one leaf in T to have degree ≥ 2 in H . Also note that v_1 and v_2 both are leaves due to the arrangement. So H has at most $k/2$ leaves and at least $(n - k)$ nodes of degree 4 or more. As $(Y-\Delta)$ -transformation keeps number of edges same. H still has at most $(2k - 3)$ edges. Thus we get, $4(n - k) + 2\frac{k}{2} + \frac{k}{2} \leq 2(2k - 3)$ i.e., $n \leq \frac{13k}{8} - \frac{3}{2}$. \square